Specialized Knowledge of Prospective Mathematics Teachers on the Concept of Mathematical Proof

Conocimiento especializado de los profesores de matemáticas en formación inicial sobre el concepto de la demostración matemática

Conhecimento especializado dos professores de matemática em formação inicial sobre o conceito de demonstração matemática

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Resumen 🔊

[Objetivo] En este artículo, se presentan los resultados de una investigación cualitativa de carácter descriptivo que tiene como objetivo caracterizar el conocimiento especializado de los profesores de matemáticas en formación inicial de la Universidad Nacional en Costa Rica (UNA), sobre el concepto de la demostración matemática, mediante el modelo Mathematics Teacher's Specialized Knowledge (MTSK). **[Metodología]** La investigación se posiciona en el paradigma interpretativo y tiene un enfoque cualitativo. Se aplicó un cuestionario, durante el primer semestre del 2021, a 42 profesores de matemáticas en formación inicial de cuarto y quinto año de la carrera Bachillerato y Licenciatura en Enseñanza de la Matemática de la Universidad Nacional de Costa Rica. Para examinar la información, se empleó el análisis de contenido y se hicieron agrupaciones de las respuestas, con el propósito de generar ideas centrales sobre el concepto de demostración. **[Resultados]** De los resultados se desprenden cuatro ideas centrales sobre lo que es una demostración matemática para los sujetos de la investigación, cercanas a aspectos formales lógico-sintácticos y matemáticos (ALSM) o aspectos informales semánticos (AIS). Se encontraron evidencias de las cinco funciones de De Villiers (1993) para la demostración y surgieron nuevas funciones relacionadas con esta en las matemáticas y en las matemáticas escolares. **[Conclusiones]** Los resultados brindan insumos a formadores de profesores de matemáticas e investigadores, en la revisión y análisis de programas de formación docente, y contribuyen en la búsqueda de nuevas áreas de investigación relacionadas con el tema.

Palabras clave: conocimiento del profesor de matemáticas; demostración matemática; concepto de demostración; formación inicial de profesores de matemáticas.

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Abstract 🔊

[Objective] This paper presents the results of a qualitative, descriptive research study characterizing the knowledge of prospective mathematics teachers at the National University of Costa Rica (UNA) concerning the concept of mathematical proof, using the Mathematics Teacher's Specialized Knowledge (MTSK) model. **[Methodology]** This research employed an interpretive paradigm and a qualitative approach. A questionnaire was administered to 42 mathematics teachers in initial training during the first semester of 2021, in the fourth and fifth years of the Bachelor's Degree program in Mathematics Teaching at the National University of Costa Rica. Content analysis was utilized to study the answers provided by participants. Groupings of answers were created to generate central ideas about the concept of proof. **[Results]** Based on the results, four central ideas were found about the nature of mathematical proof for the survey participants. These ideas are similar to the formal logical-syntactic and mathematical aspects (LSMA) or informal semantic aspects (ISA). Evidence for the five De Villiers (1993) functions of a proof was found. Moreover, new functions related to them were discovered in mathematics and in school mathematics. **[Conclusions]** The results provide input to trainers of mathematics teachers and researchers for the review and analysis of teacher training programs. Additionally, they contribute to the search for new research areas related to this subject.

Keywords: Mathematics teacher's knowledge; mathematical proof; concept of proof; prospective mathematics teachers.

Resumo 🔊

[**Objetivo**] Este artigo apresenta os resultados de uma pesquisa qualitativa descritiva que visa caracterizar o conhecimento especializado dos professores de matemática em formação inicial na Universidade Nacional da Costa Rica (UNA), sobre o conceito de prova matemática, utilizando o modelo de *Mathematics Teacher's Specialized Knowledge* (MTSK). [**Metodologia**] A pesquisa está posicionada no paradigma interpretativo e tem uma abordagem qualitativa. Foi administrado um questionário durante o primeiro semestre de 2021 para 42 professores de matemática em formação inicial no quarto e quinto ano do Bacharelado e Bacharelado em Ensino de Matemática na Universidade Nacional da Costa Rica. Para examinar as informações, foi utilizada a análise de conteúdo e as respostas foram agrupadas a fim de gerar ideias centrais sobre o conceito de demonstração. [**Resultados**] A partir dos resultados, quatro ideias centrais sobre o que é uma prova matemática para os sujeitos da pesquisa, próximas aos aspectos formais lógico-sintáticos e matemáticos (ALSM) ou aspectos semânticos informais (AIS), podem ser deduzidas. Foram encontradas evidências para as cinco funções de De Villiers (1993) para a demonstração e surgiram novas funções relacionadas à demonstração em matemática e matemática escolar. [**Conclusões**] Os resultados fornecem contribuições para educadores e pesquisadores em matemática na revisão e análise de programas de formação docente, e contribuem para a busca de novas áreas de pesquisa relacionadas ao tema.

Palavras-chave: conhecimento do professor de matemática; demonstração matemática; conceito de demonstração; formação inicial do professor de matemática.

Introduction

Mathematical proof is a practice in mathematics that has evolved in its understanding and implementation throughout its history. The modern concept of mathematical proof has been heavily influenced by symbolic logic, the foundations of mathematics (Legris, 2012), and the sociocultural context in which it has been developed (Alfaro et al., 2020).

In a study conducted by Selden and Selden (2017) on the understanding of mathematical proofs, the authors differentiated four concepts related to demonstrative activity: understanding, construction, validation, and evaluation of proofs. Mejía-Ramos et al. (2012) describe the understanding of proofs in a pragmatic way, from local to a holistic understanding of mathematical proof (Hernández-Suárez et al., 2020). However, there is no consensus among the mathematical community on a general definition of proof (Cabassut et al., 2012), resulting in a lack of clarity on what and how students understand it (Hernández-Suárez et al., 2020).

An attempt to clarify the concept of proof involves defining it in terms of mathematical logic and contrast it with what mathematicians consider to be aligned with their practical work. Thus, there are two main conceptualizations of proof: one that is linked to logic and another that is closer to the practices of mathematicians (Cabassut et al., 2012). According to Hanna and De Villiers (2012), mathematical proof can be approached in two different orientations. One orientation involves establishing a conclusion through a succession of deductive steps, where the logical and syntactic aspects of the proof take precedence. The other orientation favors the semantic components of the proof. In the latter perspective, the ideas that facilitate understanding the mathematical outcomes' validity are essential, while logical rigor is of lesser importance.

Internationally, there is consensus and wide recognition of the role of mathematical proof in training students at all educational levels, particularly, in mathematics teacher education (Cabassut et al., 2012; Mariotti, 2006; Stylianides, Stylianides, & Weber, 2017).

Conversely, the study of teacher knowledge has long been a subject of interest (Ponte and Chapman, 2006). Specifically, the works of Elbaz (1983) and Shulman (1986) during the 1980s stand out. Subsequent to these, various models have emerged to conceptualize teachers' knowledge, each one emphasizing different elements and characteristics. In these models, based on the Shulman's (1986) work, two primary components are distinguished: content knowledge for teaching and didactic content knowledge to be taught.

An especially interesting model is the one proposed by Carrillo et al. (2018): *Mathematics Teacher's Specialized Knowledge (MTSK)*. This model considers the specialized nature of the mathematics teacher's knowledge, taking into account what the teacher uses and needs, without making references to other professions. This model proposes two fundamental domains of knowledge: *(1) mathematical knowledge* that refers to what teachers know about mathematics as a scientific discipline in a school context, and *(2) didactic knowledge of the content* that refers to aspects related to mathematical content as an object of teaching and learning. It also establishes three subdomains to be considered as part of mathematical knowledge: *(a) knowledge of topics (KoT), (b) knowledge*

of mathematical structure (KMS), and *(c) knowledge of mathematical practice (KMP)*. The latter refers to different types of knowledge linked to how mathematics proceeds and is produced, that is, practices linked to mathematics; here mathematical proof takes center stage.

In this context, the study's purpose is to characterize the specialized knowledge held by prospective mathematics teachers at the National University of Costa Rica regarding the concept of mathematical proof. This concept is part of the knowledge of mathematical practice (KMP) within the MTSK model. For this purpose, two elements were particularly considered: (1) knowledge about the nature of mathematical proof and (2) knowledge about the functions of proof in mathematics and school mathematics.

Theoretical Framework

While it is indeed pertinent for a mathematics instructor to possess knowledge of the subject matter, Flores-Medrano *et al.* (2016) emphasize that mathematics teachers must also comprehend how mathematical knowledge is produced, in addition to understanding the contents and their relationships (substantive knowledge). This implies that the instructors must understand the syntactic rules of the discipline, the difference between proof, evidence, and verification, as well as the various types of proofs (syntactic knowledge). As these authors state, this allows teachers to comprehend that, in some cases, an example may correspond to a verification of a property, while in others it may correspond to a proof.

The *Mathematics Teacher's Specialized Knowledge (MTSK)* model considers the teacher's knowledge of the mathematical task as a part of mathematical knowledge and assigns, within it, a subdomain called *Knowledge of Practices in Mathematics (KPM)*. This subdomain pertains to the mathematics teacher's knowledge of how mathematics is developed, extending beyond any particular topic. Hence, the teacher's understanding of the meanings behind proofing, justifying, defining, deducing, and inducing becomes a fundamental part of the KPM (Carrillo *et al.*, 2018). Likewise, the KPM includes knowledge of the logical basis underlying each of the aforementioned practices, as well as the use and operation of the example and counterexample. As this subdomain pertains to the development of new mathematical results, i.e., the act of doing mathematics, its categories are not exhaustive and new ones may arise.

To characterize the knowledge of prospective mathematics teachers regarding the concept of proof, we used the categorization framework proposed by Alfaro *et al.* (2020) within the KPM subdomain. These authors identified several relevant elements, which must be considered in mathematical knowledge about proof. They established three components as follows:

1. *Knowledge about the nature of mathematical proof*: this component refers to knowledge about what constitutes a mathematical proof, and it includes the following subcomponents: *(a) The concept of mathematical proof:* This subcomponent is concerned with knowledge of what a mathematical proof is and what it means to prove something in mathematics. *(b) Logical validity*: It involves knowledge of how to proceed in the proof of mathematical statements, involving implicitly or explicitly the logical connectives *(and, or, or, if-then, not, among*

others) and the existential and universal quantifiers. It also includes the use of rules of inference, logical equivalences and methods of demonstration. *(c) Mathematical validity:* It refers to knowledge of rigor in mathematical proof, which implies the correct use of axioms, hypotheses, and definitions in proofs.

2. *Knowledge about the functions of proof in mathematics:* It is the knowledge about the role of proofs in mathematics and in school mathematics.

3. *The convincingness of a mathematical argument:* This refers to why mathematics teachers find a mathematical argument convincing. Subcomponents that are considered based on the work of Knuth (2002) include the use of concrete elements, familiarity**,** level of detail, ritual form, explanatory level, and validity of the argument.

In this paper, we will present the results obtained corresponding to components *(1)(a)* and *(2)*, given their close relationship to the objective of this research. Subcomponents linked to the logical or mathematical validity of the proof are not included since they could take a position, by themselves, on what it means to prove in mathematics, including logical and syntactic aspects.

Regarding the first component (knowledge about the nature of proof), specifically about the concept of proof, we consider the two conceptualizations proposed by Hanna and De Villiers (2012). The first conceptualization is related to logic, in which a formal mathematical proof consists of a succession of propositions, with the last one being the proved theorem and each of the preceding propositions being an axiom or the result of applying a rule of inference to previous propositions in the succession. In this way, the rules of inference are self-evident, and the validity of the proof can be mechanically verified (Cabassut *et al.,* 2012; Hanna and De Villiers, 2012; Tall *et al.*, 2012). For data analysis purposes, we refer to these conceptualizations as being closely related to *formal logical-syntactic and mathematical aspects (LSMA)*. The second conceptualization is linked to the practice of mathematicians, where the informal and semantic components of proof are considered. In this second conceptualization, proof serves a broader purpose than simply establishing truth. It can contribute to gaining new mathematical insights, establishing novel contextual connections, and fostering the emergence of additional problem-solving methods (Cabassut *et al.*, 2012; Hanna and De Villiers, 2012). Thus, a mathematical proof is an argument intended to convince a group of experts of the veracity of a mathematical statement and the possibility of explaining such veracity. Such proofs find their place in mathematical research journals and school and university textbooks. They typically serve as conceptual bridges in some parts of the argument, rather than an explicit logical justification. We refer to these conceptualizations as being closely related to *informal semantic aspects (ISA)*.

In relation to the second component—knowledge about the functions of proof in mathematics—this research considered the categories proposed by De Villiers (1993): *(a) verification,* proof is a way of justifying the validity of mathematical results and guaranteeing their veracity; *(b) explanation,* proof allows for deepening and understanding why a mathematical proposition is true; *(c) systematization,* proof makes it possible to logically organize a set of true statements, providing a global perspective

that favors simplification, error detection, and application in different areas, among other elements; *(d) discovery,* proof leads to finding new mathematical results in a deductive way; and *(e) communication,* proof makes it possible to share mathematical results among different actors of the scientific community, emphasizing the social process of informing and disseminating mathematical knowledge, which allows for growth, refinement, and error detection, among others.

Methodology

The study employs a qualitative approach with a descriptive scope, involving 42 prospective mathematics teachers enrolled in an Associate's and Bachelor's Degree program in Mathematics Teaching at UNA. Of the participants, 28 were enrolled in the fourth year of the program and 14 were in the first semester of the fifth year in 2021. This program is taught jointly by the School of Mathematics, which offers the mathematical component, and the Division of Educology, which offers the pedagogical component. It awards the Associate's degree, which lasts four years, and the bachelor's degree, which consists of three additional semesters and the preparation of a final graduation project.

All of the participants in the study, as part of their curriculum, have passed courses in mathematical logic and have both seen and performed demonstrations in most of the mathematics courses they have taken.

Collection of Information

A questionnaire was administered during April and May 2021, with an approximate duration of two hours. Given that the courses were held remotely, due to the COVID-19 pandemic, the questionnaire was sent to the e-mail of the participants, who completed it individually, during their classes, under the supervision of the respective teacher.

The questionnaire consists of four open-ended questions aimed at characterizing the participants' knowledge of the concept of mathematical proof. Questions 1 and 2 were oriented to define "proof", and questions 3 and 4, to its functions in mathematics and school mathematics: (1) What is mathematical proof for you? (2) What does it mean to you to prove in mathematics? (3) What is the purpose of a mathematical proof? (4) Do you consider that proof has a role in the teaching of school mathematics? If your answer is Yes, please explain its functions. If your answer is No, please explain why.

For each question, respondents were asked to provide as comprehensive and explanatory an answer as possible, supported by an example to illustrate their explanation. The purpose of the example they were asked to give was to have more elements to understand their answers and approaches. The questionnaire was shared with national and international experts in mathematics didactics, pure mathematics, and mathematics teaching, to whom the purpose of each of the questions and the categories designed for their subsequent examination were explained. They provided suggestions that enriched the instruments and refined the categories of analysis. Subsequently, everything was validated in a pilot test with prospective mathematics teachers, other than the participants considered in this study. This experience allowed for the revision and refinement of the categories of analysis.

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Analysis of Information

The information collected was examined using content analysis, which allows the coding of open-ended questions in questionnaires and the description of patterns and trends in communicative content (Cohen *et al.*, 2007). According to Cohen *et al.* (2007), this analysis involves coding, categorization (creation of meaningful categories into which the units of analysis-words, phrases, sentences-can be placed), comparison (categories and creation of links between them), and drawing theoretical conclusions from the units of analysis.

To analyze the information, each answer the participants provided was studied in joint work sessions the two researchers held. For questions 1 and 2, the conceptualizations proposed by Hanna and De Villiers (2012) were used. They can be summarized in two categories: (*a) formal logical-syntactic and mathematical aspects (LSMA)* and *(b) informal semantic aspects (ISA)*. On the one hand, in the *LSMA* category were considered those participants' answers that included, in their description, aspects alluding to sequential structures, deductive steps, mathematical theory elements (such as axioms, definitions, theorems, hypotheses, universal or existential quantifiers, logical connectors, use of inference rules, and logical equivalences, among others). On the other hand, the *ISA* category included answers that pointed more to general and not very formal aspects associated with the general argumentation for convincing, for example, if the answers contemplated the use of a drawing or manipulatives, with the aim of demonstrating or justifying a given result. It is important to clarify that one category does not necessarily exclude the other. It was possible to observe answers including aspects of both categories at the same time. In such cases, the responses were counted in both categories.

The following is an illustration, with two particular cases, of the analysis conducted for the participants' answers to question 1 of this questionnaire. Figure 1 shows the answers of participants EBM04 and ELM01, respectively, on the definition of mathematical proof.
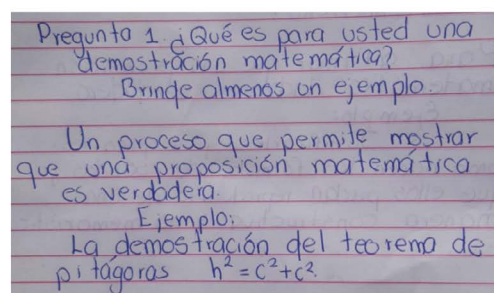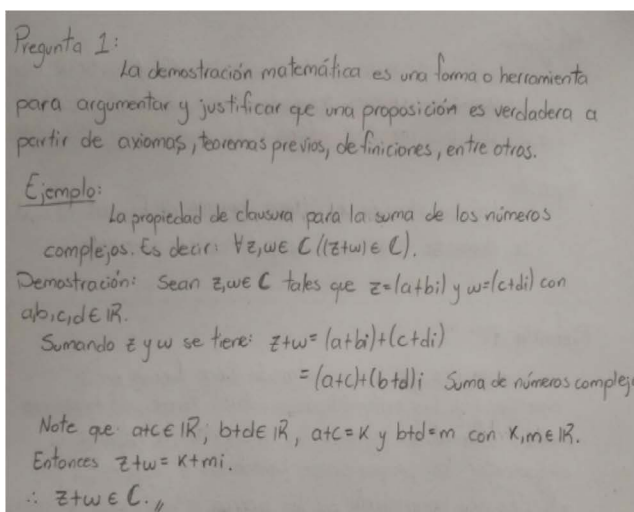

Figure 1. *Answers of participants EBM04* (left) *and ELM01* (right) *to Question 1 of the questionnaire.*

The image on the left corresponds to the answer of subject EBM04, which is located in the *LSMA* category, since it alludes to the use of formal mathematical aspects such as axioms, definitions, and theorems, among others. The image on the right shows the answer of participant ELM01 and is located in the *ISA* category, since it does not make explicit any formal logical-syntactic-mathematical element. On the contrary, it refers to processes to show that a proposition is true, in a very general way, without making formal or syntactic aspects evident for that purpose.

In the case of questions 3 and 4, to determine the functions that the participants in the study attribute to proof in mathematics and school mathematics, as a starting point, the functions used were those proposed by De Villiers (1993) and described in the theoretical framework. The answers were analyzed and placed into one of these categories. Some answers referred to only one category, while others referred to several categories at the same time. For the answers that could not be placed in the existing categories, new ones were created and specified in the results.

Results

The results were organized based on the two elements considered in this study to characterize the knowledge of prospective mathematics teachers about the concept of mathematical proof: (1) the concept of proof that refers to the knowledge about what a mathematical proof is and what it means to demonstrate a proposition in mathematics. (2) The functions of proof that refer to the knowledge about what is the role of proofs in mathematics and in school mathematics.

The Concept of Proof

In relation to knowledge about *the concept of mathematical proof*, the participants in the study offered, in question 1 of the questionnaire, a definition of mathematical proof, based on their knowledge and experiences. Most of the participants provided a definition of mathematical proof close to the category called *formal logical-syntactic and mathematical aspects (LSMA)*. Three examples of answers in this category are detailed below, corresponding to participants EBH17, EBM13, and ELH05, respectively:

- This is the evidence that a conjecture or mathematical property is true or fulfilled, based on definitions or axioms that function as hypotheses to such demonstration (EBH17).
- It is to be able to apply congruence criteria, definitions, axioms and theorems to give true results (EBM13).
- An axiomatic argumentation, sequenced and grounded in theory, to test the validity of a proposition (ELH05).

In this category, two central ideas were noticeable in the participants' answers about what a mathematical proof is.

Central idea 1 (LSMA): *mathematical proof as a noun*

Mathematical proof is associated with qualifying nouns such as a construction, a tool, a process, evidence, a mathematical reasoning, a deductive mathematical argument, and a logical and axiomatic method to determine the validity of a proposition, making use of definitions, theorems, axioms, among others.

Central idea 2 (LSMA): *mathematical proof as an action*

Mathematical proof is deemed as using or applying definitions, theorems, axioms, etc., to determine the validity of a proposition.

Both central ideas include, in their description; aspects related to the *LSMA* category. However, the answers included in *central idea 1* mainly allude to the aspects linked to mathematical logic, while the answers of *central idea 2* highlight the syntactic elements, without specifically mentioning the mathematical logic part.

Furthermore, a minority of prospective mathematics teachers defined mathematical proof more closely to the category called *informal semantic aspects (ISA)*. These definitions allude to the use of manipulatives or applications to demonstrate or convince the viewer of the results of the theorem. However, they do not demonstrate logical-syntactic or mathematical aspects. Below are the responses of participants EBH12 and ELH06 illustrating this category:

- It is a compelling mathematical argument that serves to convince that a certain mathematical idea is true, or in other words, is always fulfilled under the conditions that it involves (EBH12).
- They are a series of fundamental mathematical steps that help to explain a particular mathematical result (ELH06).

From the answers in this category, a central idea is extracted that synthesizes what a mathematical proof is for prospective mathematics teachers. For them, a mathematical proof is:

Central Idea 3 (ISA): evidence, justification, a convincing argument, a method, a process, an explanation, a series of steps to guarantee or convince about the veracity of a proposition.

Unlike *central ideas 1 and 2*, these answers do not allude to description or, in the examples provided, to elements of logical-mathematical, deductive or syntactic-mathematical logic.

Only the definition provided by participant EBH09 included in its description elements of both *LSMA* and *ISA* categories, which could be considered a central idea together with the three mentioned above.

Central Idea 4 (LSMA and ISA): a mathematical proof is a demonstration or justification by using axioms or true propositions to justify the argument to be proved. It can be supported with concrete or visual materials (EBH09).

In relation to knowledge about *what it means to prove* a proposition in mathematics, the participants' answers to question 2 of the questionnaire were grouped using the same categories used for the analysis of question 1. In contrast to question 1, only one answer was placed in the *formal logical-syntactic-mathematical aspects (LSMA)* category:

- To prove in mathematics means to argue rigorously, using theorems and axioms to construct and justify the argument. It also means using reasoning to determine the processes and resources to be used appropriately (EBM06).

This answer involves formal logical-syntactic-mathematical aspects, it refers to rigor and the need to know how to reasonably use the resources to formulate a

demonstration. However, it does not mention logical or deductive processes.

The rest of the answers were placed in the *informal semantic aspects (ISA)* category. These were grouped into three central ideas:

Proving in mathematics is an action related only to its functionality: in this case, proving refers to actions such as justifying, verifying, validating, arguing, providing evidence or substantiating the validity of a proposition. However, there is an absence of how these actions are conducted. Below are the answers of participants EBH10 and ELH11 that illustrate this idea:

- Validating an argument, a property of an algorithm, or a formula (EBH10).
- Proving in mathematics for me means verifying whether a mathematical argument is reliable to facilitate other mathematical procedures (ELH11).

Proving in mathematics is an action related to its functionality and includes the use of some tool in the process: it means to argue or validate a proposition using tools. However, the answers do not specify which tools to use or how to use them. More intuitive aspects are alluded to. The following is an example of the answer of participant EBH08:

- It means being able to use various tools of the discipline to guarantee the veracity of various theorems (EBH08).

Proving in mathematics is a skill: it refers to an ability that shows understanding of mathematics by who demonstrates and promotes criticality and abstraction to develop complex thinking. As in the previous instances, it does not lead to deductive or logical-syntactic-mathematical processes. The following example from participant ELM12 illustrates this type of answer:

- It is a skill because the meaning of the theorem has to be understood to implement, afterwards, the tools available (definitions, lemmas, etc.) and, subsequently, to build from the hypotheses and to attain the results (ELM12).

In general, for question 2, the answers of the participants in the study are oriented to explain the meaning of proving in mathematics in terms of its function. Few answers included explicit explanations or descriptions of how to proceed when performing a mathematical proof. It is understood that what was observed in the analysis of the results is being described; this does not mean that one or the other is correct or incorrect.

The Functions of Proof in Mathematics and in School Mathematics

In relation to knowledge of *the role of proof in mathematics*, as posed in question 3 of the questionnaire, evidence was found of the functions described by De Villiers (1993). Verification was the function most frequently mentioned by the participants in the study, followed by explanation, systematization, discovery, and communication.

Regarding verification, the participants pointed out that proof in mathematics serves to certify, validate, guarantee, argue, and observe the validity of a proposition and its importance in other areas. They also indicated that it serves to give meaning and congruence to mathematics and obtain the same results under similar conditions. The response from participant EBH12 is representative of this finding:

- It serves to give validity, meaning, and congruence in the construction and application of mathematics, so that, in any part of the world and in any context where mathematics is used, it has the same form, and the same results are achieved. (EBH12)

On the explanatory function, the participants consider that proof functions to determine the reasons why a proposition is true and to better explain mathematical knowledge, as well as to justify the use of a result in a given context. In this function, teachers propose an additional element that consists in the fact that proof makes it possible to debate and question a proposition with the objective of guaranteeing its validity. Two answers that illustrate the above are those of EBM02 and ELM12:

- It serves to justify to students the reasons that allow them to use a certain result in the resolution of an exercise. (EBM02)
- It serves to acquire further learning and thus be able to justify why it is necessary to implement that definition. Sometimes proof can be confusing but necessary. (ELM12)

Regarding the systematization function, the teachers under study indicate that it orders, formalizes and generalizes results, as well as identifies errors in mathematical theory. This gives foundation to mathematics as an organized theory. EBH05 provided a representative answer of this:

- In my opinion, it serves to have order and formality. Besides, if proof is not provided in mathematics, any theory or proposition might be employed

without knowing that it is not applicable only to specific cases. In other words, from proof, generalizations can be made. (EBH05)

Regarding the function of discovery, we found that proof in mathematics allows the participants in the study to construct and revise knowledge by creating new theorems and validating or discarding ideas. This broadens the contribution of existing mathematical theories in other areas of knowledge. Discovery helps to understand concrete situations and, subsequently, obtain new results or knowledge. Participant EBH09 provided insight into this function: "Proof serves to learn more about the world around us and even to learn new results or types of knowledge" (EBH09).

Participants EBM07 and EBM13 made the communication function evident as follows:

- It serves to demonstrate in a manner that anyone familiar with mathematical language is able to understand the proof. Therefore, it does not matter what language they speak, whether it is English, Spanish, or any other language. (EBM07)
- I consider that they serve to show truths, debate and question. I see this when I am trying to solve an exercise that I have to present. I always act as if I were marketing a product; I try to make people who read my proof buy it, to build up their confidence to go ahead with the purchase. This is how I see the usefulness of proof, I sell useful knowledge. (EBM13)

In addition to the five functions mentioned above, another emerged from the

answers of the participants; it is related to the development of skills. Here is participant ELM02 said:

- Mathematical proofs serve to develop skills such as critical and deductive thinking, argumentation, among others. They allow justifying mathematical procedures with logic. (ELM02).

Regarding knowledge about *the role of proof in school mathematics*, question 4 in the questionnaire, a few participants responded that mathematical proof has no role in this context. They argue that while mathematics teachers should know how to prove, in their experience teaching school mathematics, they have not had occasion to utilize proof. The following are two answers that support these results:

- If we focus on school teaching, it is important that teachers know how to demonstrate, however, when a subject is going to be taught in the classroom, it is not going to be through proofs. (EBM01)
- Maybe for you who train future teachers it does work, you learn from it somehow; but, if you go to a school, what do you need it for? I have been tutoring or giving private classes for years and, at no time, have I ever used it. (EBH10)

Most participants ascribe one or more roles to proof in the teaching of school mathematics. This study identified three of the functions of mathematical proof proposed by De Villiers (1993): verification, explanation, and discovery. Additionally, three emergent functions associated with proof in school mathematics were identified: the

development of skills in students, the contribution to students' affective mastery, and the contribution to teaching practice.

Regarding verification and discovery functions, participants find mathematical proof useful in school mathematics for validating results and facilitating the construction of new knowledge. This supports students' understanding that mathematics is formal. Besides this, the explanatory function aims to provide meaning to school mathematical knowledge. It also helps students understand why a mathematical proposition is true, know its origin, usefulness, and foundations, and deepen their understanding of the topics.

The following is the answer from participant ELM04, which serves as evidence for verification, as well as that of ELM12, who refers to discovery and explanation:

- It helps students verify mathematical concepts or formulas. (ELM04)
- It serves to build knowledge and, thus, learn how and why certain characteristics or theorems are fulfilled. (ELM12)

As for the function associated with the development of skills in students, the participants in the study mentioned that it fosters or develops argumentation, mathematical reasoning and thinking, as well as critical, logical, and deductive thinking, formulation and validation of conjectures, and problem solving. The answers provided below are from participants EBM07 and EBM18, referencing the mentioned function:

- To develop mathematical reasoning and thinking in students. (EBM07)
- From a very early age it would be good to teach children to think in a

non-mechanical way, but rather to teach them critical, logical, and deductive thinking; a thinking process that, I believe, proof develops. (EBH18)

The function associated with the contribution to the affective domain in students is related to aspects of students' mathematical identity, motivation, love, and interest in mathematics, curiosity, self-confidence, and familiarity with mathematics. EBM03 and ELM01 provided the following answers that highlighted the function:

- To motivate to learn about new propositions. (EBM03)
- To help awaken curiosity in students, and also to trust the contents being covered in class, not because the teacher says so, but because there is mathematical validity. (ELM01)

The function associated with contributing to teaching practice involves the utilization of proof to enable mathematics teachers to know the fundamentals of school mathematical content. It also serves to justify the validity of a proposition if questioned by students. In addition, it provides teachers with procedures or arguments for teaching mathematical knowledge. ELM03's answer illustrates the above:

- As a teacher, [it is important] to know the origin of different topics you have to teach, that is, to have specialized knowledge. However, using this knowledge to explain topics might appear somewhat abstract to students. It would be used if students want to know that what is explained was not made up, but that it is true. (ELM03)

Conclusions

The study aimed to characterize the specialized knowledge of prospective mathematics teachers at the National University of Costa Rica about the concept of proof as part of the *KMP* of the *MTSK* model.

The results of this work allowed the researchers to provide a characterization of the knowledge of prospective teachers at the UNA on the concept of mathematical proof, in which the logical-syntactic aspects become relevant, without neglecting the informal and semantic aspects.

Regarding the question of *what a mathematical proof is*, teachers in initial training provided a definition that includes *formal logical-syntactic and mathematical aspects (LSMA)* in their descriptions. The definition refers to deductive or sequential processes in which theorems, axioms, definitions, among others, are used. Two trends of answers were observed, synthesized as central ideas and intended to gather the feelings of the participants: mathematical proof defined as a noun and as an action.

The teachers who provided a definition close to *informal semantic aspects (ISA)* mentioned the use of manipulatives, drawings, explanations or applications to demonstrate or convince learners of the results, without referring to logical-syntactic or mathematical aspects.

We found definitions of mathematical proof that included elements of *LSMA* and *ISA* categories, which referred to both logical-syntactic and mathematical as well as semantic aspects.

Regarding *what proving a proposition means* in mathematics, the results are the reverse of the above. Teachers in initial training associate proof with *informal semantic aspects (ISA)*. For them, proving in

mathematics is an action related only to its functionality. This means justifying, verifying, validating, arguing, evidencing or substantiating the validity of a proposition. It is also an action linked to its functionality and includes, in the process, the use of some tool of the discipline, such as definitions, axioms, theorems, symbols, language, among others. Finally, it is a skill.

On the *functions of proof in mathematics*, the participants mentioned all five of De Villiers' (1993) functions, with a significant inclination towards verification and explanation. From the functions mentioned by prospective mathematics teachers, a new function emerges: proof for skill development. This function is the most attributed to proof in school mathematics, followed by explanation. The participants in the study affirmed that proof in school mathematics serves to foster skills in the student, such as argumentation, mathematical reasoning and thinking, critical, logical, and deductive thinking, formulation of conjectures, problem solving, motivation, love and interest in mathematics, curiosity, confidence, and familiarity with mathematics. In addition, they help to give meaning to school mathematical knowledge. From the functions proposed by De Villiers (1993) for proof in school mathematics, only three were found: verification, explanation, and discovery.

It is important to clarify that, in this study, there is no consideration of correct or incorrect answers, nor are the results being generalized. The aim is to characterize the knowledge about proof, specifically, to know what it is, what it means, and what proof is for according to prospective mathematics teachers, thus, complementing the work done by Alfaro *et al.* (2020), in which the research focus was the knowledge of teachers in initial training about logical validity (see component 1b of the theoretical framework) and how to proceed in the demonstration of mathematical statements. The categories of analysis were precise and clearly defined *a priori* for this study since the knowledge to be characterized was more explicit and based on mathematical knowledge.

In turn, our findings are nourished by the categorization proposed by Alfaro *et al.* (2020) and contribute to other categories of the model not explored by those authors, related to the concept of proof itself, its meaning, and its functions.

In this investigation, since no *a priori* position on the concept of proof is assumed, the categories of analysis are emergent, unlike the studies of Alfaro *et al.* (2020) in which these categories were clearly identified and substantiated on mathematical logic.

Likewise, this work could provide input to mathematics teacher trainers and researchers in the revision and analysis of initial and continuous teacher training programs in mathematics teaching, as well as contribute to the development of new research areas related to mathematical proof. Inquiring about what the function of proof is for future teachers, what it means, and what its function is helps to elucidate what perceptions are being generated in them and, with this, to make curricular and methodological decisions in response to the results.

It is evident that there is a need for mathematical teacher training programs to discuss and reflect on proof and its functions as a possible object of study and not only as a tool in mathematics courses.

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Informed Consent

The authors declare that the participants in this study were informed about the treatment of the information.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Author Contributions Statement

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Data Availability Statement

Data supporting the results of this study will be made available by the corresponding author [**C.A.C.**] upon reasonable request.

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